MATC44 Week 3 Notes

1. Principle of Extremals:

- If we want to show a certain construction exists, then we consider the largest or smallest among a specific class of structures. The largest or smallest one might be the one that satisfies the property/ies.

- If we want to show a certain construction can't exist, then, arguing by contradiction, we assume that at least one such construction exists. Next, we consider the largest or smallest such construction and using it and the assumptions of the problem, we deduce that there is an even larger or smaller, respectively, construction. But, this contradicts the fact that we started with the largest or smallest, respectively, construction. Hence, no such construction can exist.

2. Examples:

a) Consider 100 numbers, $a_1, a_2, \ldots, a_{100}$, each on the vertex of a regular 100-gon. We assume that any number is the avg of its two neighbours.
I.e. $a_i = \frac{a_{i-1} + a_{i+1}}{2}$

Show that all these numbers are equal. I.e. $a_1 = a_2 = \ldots = a_{100}$

Soln:
Consider the largest number of these 100 numbers, a_j. From the assumption, a_j is the average of its 2 neighbours, a_{j-1} and a_{j+1}. However, since a_j is the largest number, that means a_{j-1} and a_{j+1} must equal to a_j. I.e, $a_{j-1} = a_j = a_{j+1}$. Therefore, a_{j-1} and a_{j+1} are also the greatest numbers. So, by the above reasoning, they are also equal to their neighbours. We proceed inductively.

b) Consider n points on the plane. Each point is either red or blue. Assume that on any edge with endpoints of the same colour there is a point of diff colour. Show that all points must lie on the same line.

Soln:
Assume that not all points are on the same line. Then, we can form triangles using our points. We consider that triangle with the smallest area. Let it be ABC. 2 of its vertices will have the same color. Hence, by assumption, there will be another point, D, of a diff colour on the edge btwn the 2 points with the same colour. However, the new triangle has a smaller area than the original area, which is a contradiction. Hence, there cannot be any triangles and hence all points lie on the same line.

c) Show that $\sqrt{2}$ is irrational.

Soln: Assume that $\sqrt{2}$ is rational.
Then, $\sqrt{2} = \frac{P}{Q}$, $P, Q \in \mathbb{Z}^+$

Furthermore, there is a multiple of $\sqrt{2}$ which is a natural number.
$A = \{n \in \mathbb{N} \mid n\sqrt{2} \text{ is a natural num}\}$
By assumption the set A is a non-empty subset of the natural nums. Hence, it must have a smallest element, n_0. Let $n_1 = n_0\sqrt{2} - n_0$. Thus, n_1 is a natural num. $(n_1)\sqrt{2} = 2n_0 - \sqrt{2}n_0$. n_1 is still a natural num. Furthermore,
$1 < 2 < 4$
$1 < \sqrt{2} < 2$

$$(n_1)\sqrt{2} = (n_0)(2-\sqrt{2})$$
$$n_1 = \frac{(n_0)(2-\sqrt{2})}{\sqrt{2}}$$
$$< n_0$$

However, this is a contradiction to our assumption that n_0 is the smallest element in A.
$\therefore \sqrt{2}$ is irrational.

3. Erdös - Szekeres Thm:
- The Erdös-Szekeres Thm concerns with maximally ordered numbers in a random list of nums.
- Thm: Consider n^2+1 real numbers $a_1, a_2, \ldots, a_{n^2+1}$. Show that there is always an inc or dec sub-sequence consisting of $n+1$ numbers.
- Definitions:
  a) Sequence: An enumerated collection of objs in which order matters. I.e. $1,3,5,6 \neq 1,5,3,6$
  b) Sub-Sequence: What remains after we remove some terms from a seq. I.e. $1,3$ is a sub-seq of $1,3,5,6$.
  c) Increasing Sequence: A seq is inc if each term is $\geq$ than the prev term. I.e. $x_1 \leq x_2 \leq, \ldots, \leq x_n$
  d) Decreasing Sequence: A seq is dec if each term is $\leq$ than the prev term. I.e. $x_1 \geq x_2 \geq, \ldots, \geq x_n$

- Proof:

Let's consider a few special cases first:

a) $n=1$: Then, we are given $2(1^2+1)$ numbers, say a_1 and a_2. Clearly, we will have either $a_1 \le a_2$ or $a_2 \le a_1$.

b) $n=2$: Then, we are given $5(2^2+1)$ nums, a_1, a_2, a_3, a_4, a_5. We want to find $3\ (2+1)$ of them which form an inc or dec chain. Assume that there is no list of 3 of them that is inc. We will then show that there are 3 of them which form a dec list.

Consider the longest dec list starting with a_1. If this list has at least 3 nums, then we are done. Assume that this list has 1 or 2 nums.

Consider the longest dec list starting with a_2. If this list has at least 3 nums, then we're done. Assume that there are only 1 or 2 nums in this list.

Consider the longest dec list starting with a_3. If this list has 3 nums, then we're done. Assume that there are only 1 or 2 nums in this list.

Consider the longest dec list starting at a_4. This list has at most 2 nums, a_4, a_5.

Consider the longest dec list starting at a_5. This list has exactly 1 num, a_5.

Hence, we have 5 nums and for each of them we consider the length of the longest dec seq starting with them. The lengths are either 1 or 2. Hence, we have 5 numbers and for each of them, we have a list of either 1 or 2 nums. By P.P, there must be 3 nums, x_1, x_2, x_3, that have the maximal associated dec seq of the same length (1 or 2). Say that this length is 2. We claim that $x_1 \leq x_2 \leq x_3$. To show this, assume that $x_1 > x_2$. Then, by considering the longest dec list of 2 nums starting with x_2 and joining to this list x_1, we would obtain a dec list of 3 nums which start from x_1. This is a contradiction to our assumption and $x_1 \leq x_2$. Similarly, we show that $x_2 \leq x_3$. Hence, we have $x_1 \leq x_2 \leq x_3$, which is a list of 3 nums which are inc. $\therefore$ There is always a list of 3 nums which is either inc or dec.

General Case:
Let $n \in \mathbb{N}$. Consider for each num a_i the longest dec seq starting from a_i, $i=1, \dots, n^2+1$. This list will contain l_i nums.
If $l_i \geq n+1$, for some i, then we are done.
If $l_i < n+1$, for all i, then we have $1 \leq l_i \leq n$.
Hence, we have n^2+1 numbers l_i that can take at most n values. By P.P, there must be $n+1$ nums l_i which are equal.
Let these nums be $l_{k_1}, \dots, l_{k_{n+1}}$.
Consider the associated nums from our list $a_{k_1}, \dots, a_{k_{n+1}}$. Then, all these nums have the property that the longest dec list starting with each of them has the same number of numbers for all of them. This implies that these numbers must form an inc seq $a_{k_1} \leq a_{k_2} \leq \dots \leq a_{k_{n+1}}$. Hence, we have a list of $n+1$ nums that is inc.

4. Diophantine Approximations:
- Thm: Let a be a positive irrational num and let $\varepsilon > 0$ be any positive num. Then, there are natural nums n, m s.t. $|n \cdot a - m| \leq \varepsilon$.
- Corollary: There are natural nums n, m s.t.

$$\left| a - \frac{m}{n} \right| \leq \frac{\varepsilon}{n} < \varepsilon.$$

– Proof: First of all, it's easy to find infinitely many m, n s.t.

$$\left| a - \frac{m}{n} \right| \leq \frac{1}{n}$$

Note that the above inequality is equal to $|n \cdot a - m| \leq 1$.
Consider now any natural num $n > \frac{1}{a}$ and consider the num $n \cdot a$. Clearly, this num must be between 2 consecutive natural nums m and $m+1$. Then, for this m we have $m \leq n \cdot a \leq m+1$ which implies that $|n \cdot a - m| \leq 1$. If $\varepsilon > 1$, then we are done.
If $\varepsilon < 1$, then we consider the multiples of ε: $0 < \varepsilon < 2\varepsilon < \cdots < (k-1)\varepsilon < 1 < k\varepsilon$
Hence, the multiples of ε divide the interval $[0,1]$ in k intervals:

1. $[0, \varepsilon]$,
2. $[\varepsilon, 2\varepsilon]$,
3. $[2\varepsilon, 3\varepsilon]$,

$\vdots$

k. $[(k-1)\varepsilon, k\varepsilon]$

Let's now consider $k+1$ diff nums of the form $n \cdot a - m$ which are less than 1. Since we have $k+1$ nums in a set of k intervals, 2 of them must be in the same set. But their difference is bounded by ε, since ε is the size of the interval.
The difference of nums of the form $n \cdot a - m$ is of the same form. Hence, we proved that there exists nums m, n s.t. $|n \cdot a - m| \leq \varepsilon$ and $\left| a - \frac{m}{n} \right| \leq \frac{\varepsilon}{n} \leq \varepsilon$

– E.g. Suppose that there is a power of 2 that starts with 2017.
I.e. There is $n \in \mathbb{N}$ s.t. $2^n = 2017\ldots$
Prove that there are natural nums m, n s.t.
$2017 \cdot 10^m \leq 2^n \leq 2018 \cdot 10^m$

Soln:
Let $a = \log 2$. Then we get $\log(2017) + m \leq n \log 2 \leq \log(2018) + m$, or $\log 2017 \leq n \log(2) - m \leq \log 2018$.

I.e. If $a = \log 2$
$2017 \cdot 10^m \leq 2^n \leq 2018 \cdot 10^m$
$\rightarrow \log(2017) + m \leq n \log 2 \leq \log(2018) + m$
$\rightarrow \log 2017 \leq n \log(2) - m \leq \log(2018)$

Then, there are natural numbers m', n' s.t.
$$|n' \log(2) - m'| \leq \frac{\log(2018) - \log(2017)}{2}$$

Then, an appropriate multiple of the num $n' \log(2) - m'$ will lie between $\log 2018$ and $\log 2017$. This multiple gives us the desired result.